Exercises

Derivatives and extrema of multivariate functions

Exercise 1. Choose *one or two* of the following functions and compute its gradient.

1. $f(x_1, x_2, x_3) = \sqrt{x_1^2 + x_2^2 + x_3^2}$
2. $f(x_1, x_2, x_3) = \sin x_1 \cdot \cos x_2 \cdot \sqrt{x_3}$
3. $f(x_1, x_2, x_3) = 2x_1^2 \ln(x_2^2) + e^{x_3^2} \cdot \sin x_1$
4. $f(x_1, x_2, x_3) = 3x_1^2 x_2 + 4x_2 x_3^2 + x_1 x_2 x_3$

Exercise 2. Compute the gradients and the Hessian Matrices (i.e. all partial derivatives of order 1 or order 2) of *one* of the following functions.

1. $f(x_1, x_2, x_3) = x_1^{\ln(x_2^2 + x_3)}$
2. $f(x_1, x_2) = x_1^2 x_2 + e^{x_1} x_2$.

Exercise 3. A manufacturer produces two goods with the demands

$$x_1(p_1, p_2) = 10 - p_1 + 2p_2 \text{ and } x_2(p_1, p_2) = 8 + 2p_1 - 6p_2$$

depending on the prices p_1, p_2 of the two goods. The production cost of good 1 is 4, the production cost of good 2 is 2.
Compute the prices such that the manufacturer maximizes his profit.

Exercise 4. Compute the local extrema of *one* of the following functions f:

1. $f(x, y) = e^{x^2 - 4x} + \frac{1}{3}y^2 - 2y$
2. $f(x, y) = x^3 - y^3 + 27x + 12y - 4$
3. $f(x, y) = x^3 - y^3 - 27x + 12y - 4$

Exercise 5. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ with $f(x, y) = x^2 \cdot e^{y^2}$.

1. Compute the gradient ∇f of the function. In particular compute the gradient in the point $(x_0, y_0) = (1, 0)$.

2. The **total differential** is defined as $df := \sum_{i=1}^{n} \frac{\partial}{\partial x_i} f$ (i.e. the sum of all partial derivatives). By setting $df(\Delta x) := \sum_{i=1}^{n} \frac{\partial}{\partial x_i} f \cdot \Delta x_i$ we can approximate the function at $x^{(0)} + \Delta x$ with

$$f(x) = f(x^{(0)}) + df(\Delta x, x^{(0)}).$$

   (a) Compute the total differential of f at $x^{(0)} = (1, 0)$.

   (b) Approximate $f(1.1, 0.1)$ with the total differential.

Exercise 6. If the partial derivatives of a function f are continuous the **directional derivative** at some $x^{(0)}$ for some direction $v \in \mathbb{R}^n$ is defined as

$$D_v f(x^{(0)}) := \frac{1}{\|v\|} \sum_{i=1}^{n} \frac{\partial}{\partial x_i} f(x^{(0)}) \cdot v_i.$$

For the function $f(x_1, x_2, x_3) = x_1 x_2^2 + x_2 \ln x_3$ compute the directional derivative $D_v f(x^0)$ im Punkt $x^0 = (1, 3, 5)$ für $v^{(1)} = (1, 3, 4)$ und $v^{(2)} = (0, 1, 0)$.